

Scaling in a simple model for surface growth in a random medium

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Surface growth in random media is usually governed by both the surface tension and the random local forces. Simulations on lattices mimic the former by imposing a maximum gradient m on the surface heights, and the latter by site-dependent random growth probabilities. Here we consider the limit $m \rightarrow \infty$, where the surface grows at the site with minimal random number, *independent* of its neighbors. The resulting height distribution obeys a simple scaling law, which is destroyed when local surface tension is included. Our model is equivalent to Yee's simplification of the Bak-Sneppen model for the extinction of biological species, where the height represents the number of times a biological species is exchanged.

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Invasion percolation [1] is a model for viscous fingering, in which at each time step a segment of the interface moves into the capillary channel which imposes the (overall) minimum resistance. Numerically, this is modeled by a lattice of sites, where each site is assigned a random number between zero and one, and by moving the interface into the perimeter site with the smallest random number. After a while, most of the perimeter random numbers are relatively large (above the percolation threshold), and growth occurs in “bursts” which explore the vicinity of where they started [2].

Another family of growth models is based on the “solid on solid” concept, where one ignores overhangs and allows only steps which increase the height h of the interface (relative to the initial line or plane). Such models, which also allow for surface tension, include the Kardar-Parisi-Zhang (KPZ) [3] model, where the random resistance depends on horizontal position x and on time, and the Parisi [4] model, where this resistance is local, depending on both x and h . A discrete version of the latter, by Kim and Kosterlitz [5], only allows steps which obey the constraint $|h(x) + 1 - h(x \pm 1)| \leq m$, with $m = 1$. Sneppen (and Jensen) [6] considered variants of this model, in which neighbors continue to be updated until the Kim-Kosterlitz constraint is obeyed everywhere. Bak and Sneppen [7] then used a related model to describe the extinction of biological species. In their model, they always updated also the nearest neighbors of the growing site, irrespective of the height gradients. However, unlike Sneppen and Jensen, they did not continue these updates in an “avalanche” beyond the nearest neighbors. The “bursts” were thus more localized than in the original Sneppen model.

Recently, Yee [8] introduced a simplified version of the Bak-Sneppen model, which can be identified as the $m \rightarrow \infty$ limit. In this model, there is no surface tension constraint on the growth, and thus each

“pillar” continues to grow as long as the random number in front of it is smaller than those facing all other pillars. Although this model preserves the long range information on all the perimeter “resistances”, it omits the interactions between neighbors, and is thus geometry and dimension-independent. In the language of invasion percolation, this model could describe a collection of independent capillary channels, penetrated from one end by a viscous fluid from the *same* reservoir, and with independent randomly varying resistances along each channel. The removal of interactions allowed an analytic solution of many aspects of this model [9], in agreement with simulations, which were phrased in the biological language. In the present paper we re-interpret this model as one of surface growth, and test if the resulting height distribution of this growth obeys a standard scaling law. We start with the definition of the model and its simulation and end with a partial analytical treatment.

The Yee version of the Bak-Sneppen model takes an array of L random numbers r_i , $i = 1, 2, \dots, L$ initially distributed between zero and unity. At each time step $t \rightarrow t + 1$, the smallest of the L random numbers is replaced by a new random number. After some time [9], nearly all random numbers are very close to unity, and growth occurs in large “bursts” where a single “pillar” grows, similar to invasion percolation [2]. This version of the model [8] is independent of any geometry; the full Bak-Sneppen model replaces also the lattice neighbors of the lowest random number, and thus depends on the assumed lattice geometry; it gives a threshold x_c below unity such that after a long time $x_c < r_i < 1$ for nearly all i . The random numbers r_i can be interpreted as fitness [7] in biological evolution, or as the quality of court decisions in a judicial system with law-by-precedent [8]. We now use the physical interpretation [6].

With every element i we associate a height variable h_i which initially is zero for all i , and then increases by unity every time this element i gets a new random value r_i (which corresponds to the new perimeter site in front of the moving interface). We can imagine a deposition process in which bricks drop down onto the site with the lowest random number r_i and then change this r_i into another random number, or as the viscous fluid moving along the i 'th (one dimensional) capillary channel, and reaching a new resistance. What is the probability distribution function (proportional to the histogram) of the observed heights? For this purpose we stop the growth whenever the highest h_i value reaches a predetermined value L_z (as usual in invasion percolation simulations and experiments [2]).

To avoid overcrowding of our figures we binned the observed heights into powers of two; that means the k -th bin contained heights between 2^{k-1} and $2^k - 1$. It is plausible that for $1 \ll h_i \leq L_z$ and large L_z the results should depend mainly on the ratio h_i/L_z . Figure 1 shows in its three parts the binned histogram $N(h)$ in the scaled form N/L_z versus h/L_z , giving a good data collapse for large enough L ; the three parts correspond to $L_z/L = 0.1, 1$ and 10 and for large L seem to give the same curve. The initial linear increase, for $h \ll L$, in these log-log plots, together with the fact that the bin size increases as the height h , would imply that without binning this increase corresponds to a constant probability distribution function $\propto N$. At large heights we see a cut-off since $h > L_z$ is impossible. In fact, Fig. 2 shows that the unbinned distribution follows roughly an exponential, $\propto \exp(-6h/L)$.

The time τ after which the tallest pillar hits the top, $h_i = L$, is shown in Fig. 3 for our squares (and rectangles). It increases roughly as L^z , $z \simeq 1.8$. However, a slight curvature suggests that

asymptotically the exponent z may be 2. Indeed, a plot of τ/L^2 versus $1/L^{0.3}$ (Fig. 4) seems to approach a finite limit for $L \rightarrow \infty$. This means that when the tallest pillar hits the top, a finite fraction of the whole $L \times L$ lattice is occupied by the bricks of the pillars (or by the invading fluid). About the same intercept 0.07 was also found from the high and the flat rectangles of Fig. 1, for $\tau/(L_z L)$ versus $1/L^{0.3}$.

As a function of time t , the average height H trivially always increases linearly in time, while the width $W \propto t^\beta$ of the height distribution has an exponent β increasing towards unity for t increasing towards τ (not shown).

Now an interaction between neighboring sites i is introduced. If i is updated, then also $i + 1$ and $i - 1$ are updated if $|h_i - h_{i+1}|$ or $|h_i - h_{i-1}|$, respectively, are $\geq m$. Here m is a fixed number between 0 and L_z . This interaction corresponds to some sort of surface tension, which tries to avoid too large gradients in the height profile h_i . It also makes our model one-dimensional, since now the neighborhood introduces a geometry. The limit $m = 0$ corresponds to the Bak-Sneppen model (always updating of neighbors) and the limit $m = L_z$ to the simplified Yee version (no updating of neighbors).

Figure 5 shows how the time τ , the average height $H = \langle h_i \rangle_i$ and the surface roughness $\langle (h_i - H)^2 \rangle^{1/2}$ depend on this new parameter m ; they go neither to infinity nor to zero, but the height H has a pronounced minimum at small m . Because of this new length m , the above simple scaling in terms of h/L_z no longer works, even if as in Fig. 6 we take m to be that value (5 to 20) for which H has a minimum. Thus the non-interacting version obeys simple scaling while the interacting version depends on the geometry (here: one-dimensional only) and disobeys simple scaling.

As stated, many features of the Yee model were calculated analytically by Newman [9]. Specifically, at time t the smallest random number on the “perimeter” was shown to grow as

$$x(t) = \frac{t}{t + L} \quad , \quad (1)$$

approaching unity at long times. Indeed, if a “pillar” starts growing at a time $t \gg L$ then it will continue to grow for n consecutive steps, during which the new random numbers encountered by this “pillar” are smaller than $x(t)$. The probability of such an n -step growth was found to be exponential:

$$p_n = x^{n-1}(1 - x) \quad , \quad (2)$$

yielding an average step of length

$$\langle n \rangle = \frac{1}{1 - x} = \frac{t + L}{L} \quad . \quad (3)$$

Indeed, our simulations confirm “bursts” whose length grows linearly with time. Also, the same formalism yields

$$\langle (n - \langle n \rangle)^2 \rangle = \frac{t(t + L)}{L^2} \quad \text{or} \quad \langle (n - \langle n \rangle)^2 \rangle / \langle n \rangle^2 = \frac{t}{t + L} \quad (4)$$

which also agrees with our simulations for large L and t , see Fig. 7. This unifractal distribution, in which n scales (for $1 \ll L \ll t$) as $n \sim t/L$, is clearly different from that expected in other growth models mentioned in our introduction.

Note also that a given growth stops when the next random number is larger than x , which happens with probability $1-x = L/(L+t)$. Assuming that the random numbers are distributed equally between zero and one, every site will be encountered at least once when $1-x$ becomes smaller than $1/L$, i. e. at times of order L^2 . It is not clear yet if this result relates to our numerical values for τ , which were asymptotically consistent with being $\propto L \times L_z \propto L^2$.

Ref. [9] also derived the probability to find a growth “run” of length n at *any* time,

$$P_n = \sum_{t=0}^{\infty} x(t)^{n-1} (1-x(t))^2 \propto \frac{1}{n} \quad . \quad (5)$$

The final height h_i of a “pillar” is a sum over such “run” lengths, which grow longer and longer with time. Our simulations show that for $L_z = L$, the typical number of such runs is of order 3–5. Thus, the distribution $N(h)$ discussed above should in principle be a convolution of distributions like N_n . We evaluated several such convolutions, with a variable number of “runs”, and they all seem to converge asymptotically (for large h) to $N(h) \sim 1/h$, which differs from the exponential form found numerically in Fig. 2. At the moment we have no explanation for this discrepancy.

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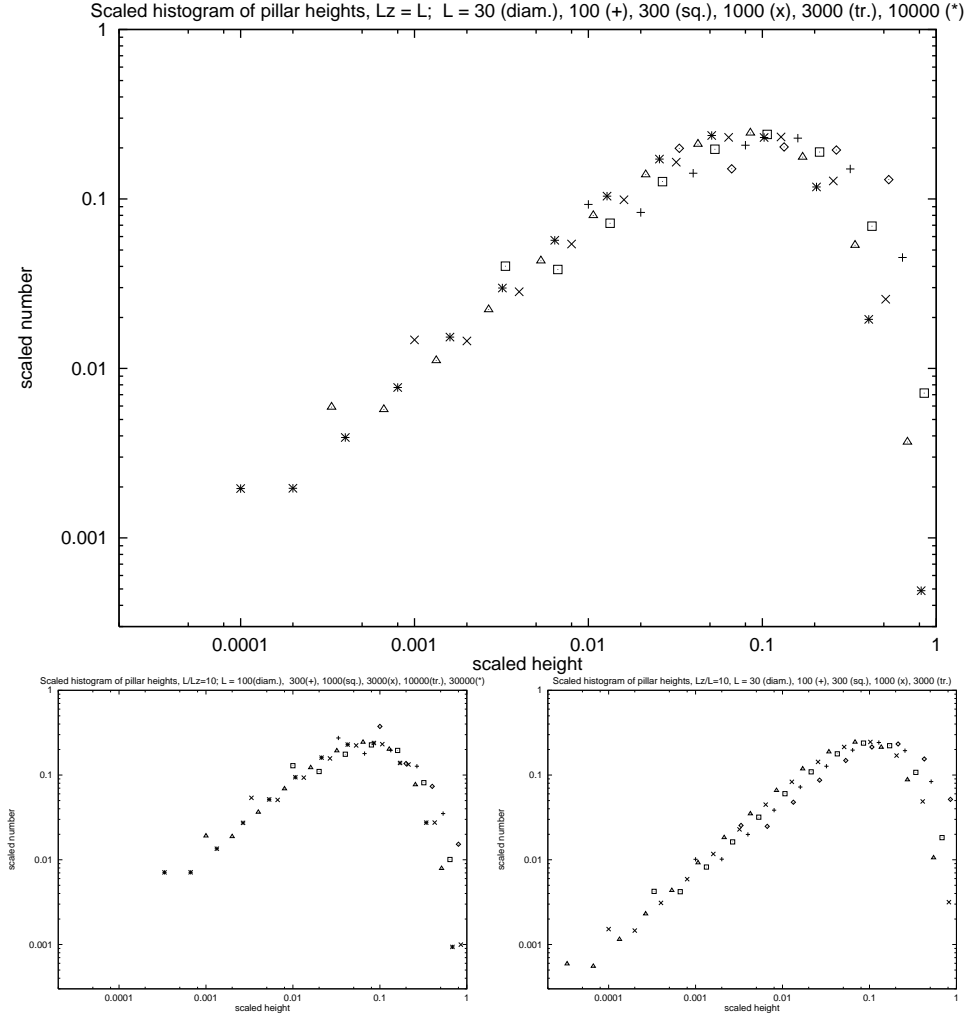


Figure 1: Scaling of the pillar height histograms: Curves for different L collapse, except for very small and very large heights. The shape of the $L \times L_z$ lattice is a square (part a), a flat rectangle (part b), and a high rectangle (part c). Up to 640,000 samples were averaged over.

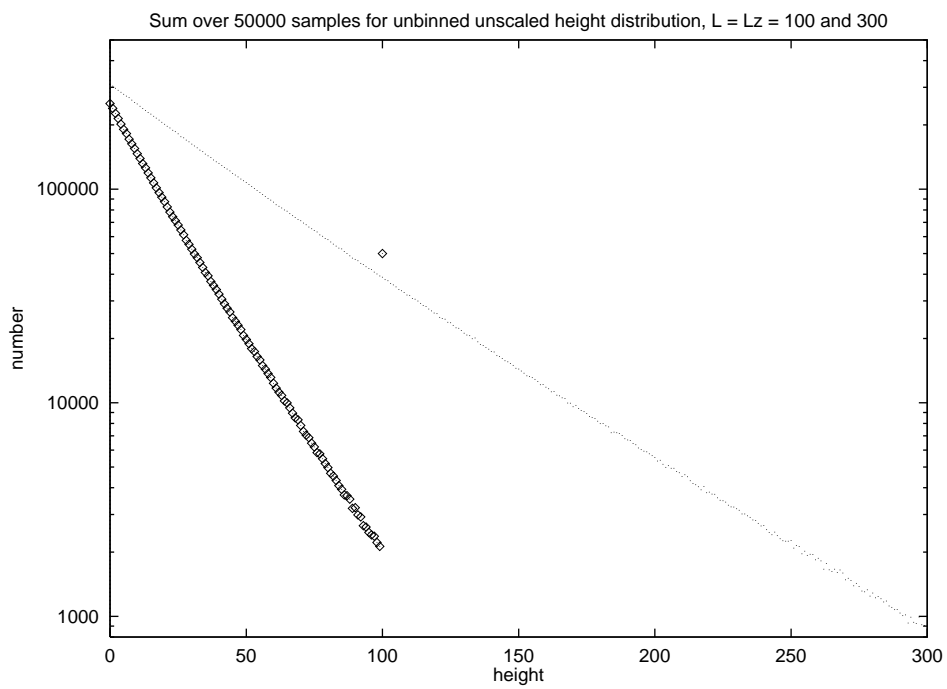


Figure 2: Examples of unbinned and unscaled distributions of heights, showing a roughly exponential decay, for $L = L_z = 100$ and 300 .

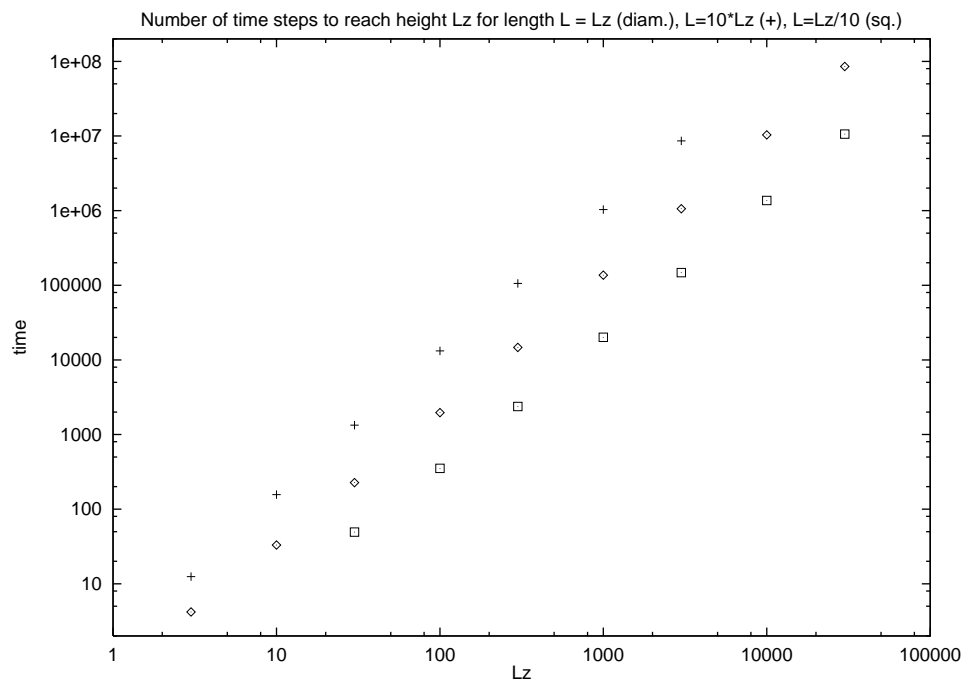


Figure 3: Time to reach the top, versus L , for squares, flat, and high rectangles.

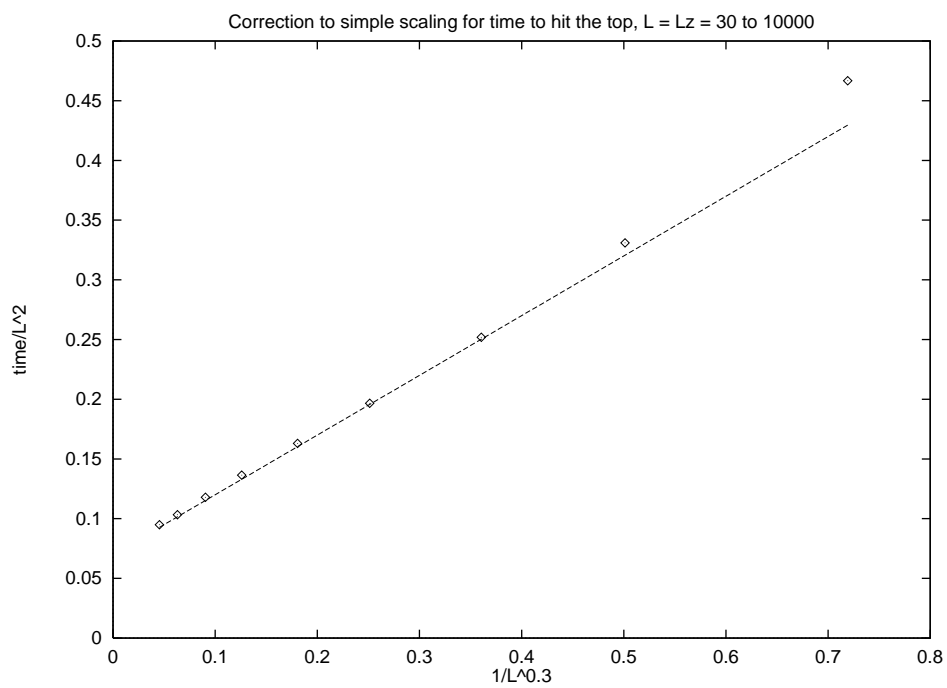


Figure 4: The times of Fig. 3 are shown to follow $\tau/L^2 = 0.07 + 0.5/L^{0.3} + \dots$ for large systems.

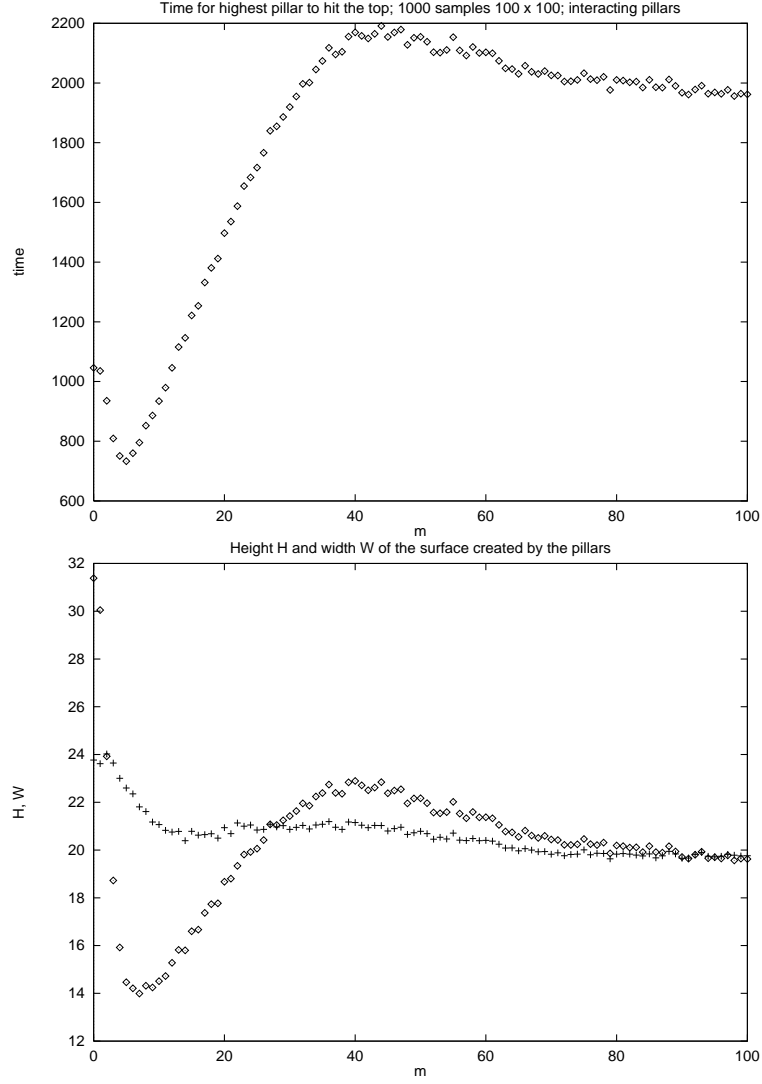


Figure 5: Influence of interaction parameter m at $L = L_z = 100$, averaged over 1000 samples, with $m = L$ corresponding to the simplified Yee model and $m = 0$ to the one-dimensional Bak-Sneppen model. Part a shows the time to reach the top, part b the height (diamonds) and the width (+) of the surface defined by the pillar tops.

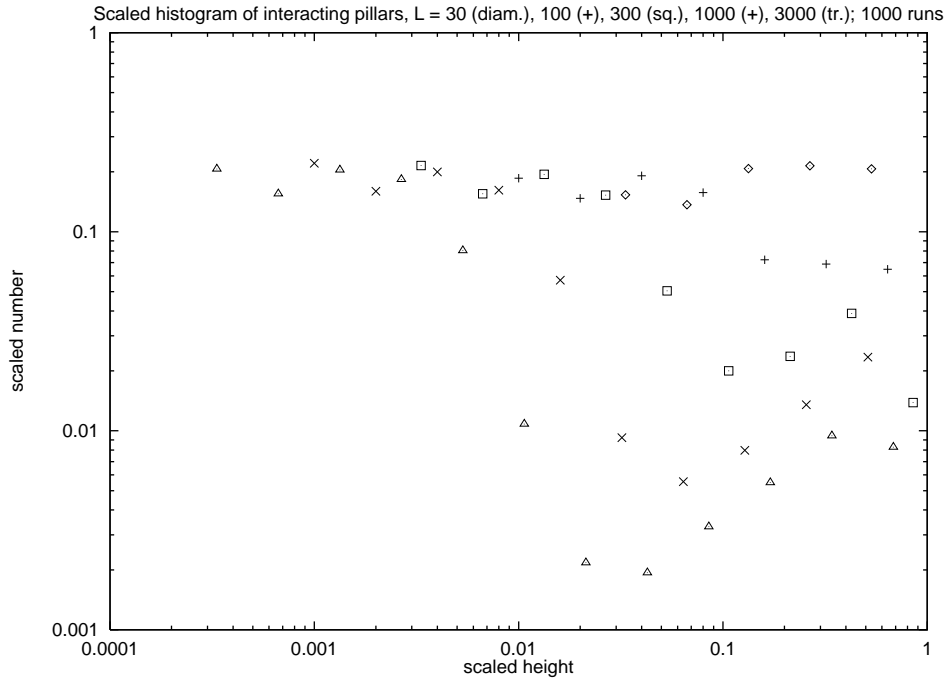


Figure 6: Failed scaling of pillar height histograms for thousand $L \times L$ samples. The interaction parameter m was taken to give a minimum of $H(m)$ (diamonds in Fig. 5b) and varies from 5 to 20. In contrast to the analogous Fig. 1, the different curves do not collapse to one curve.

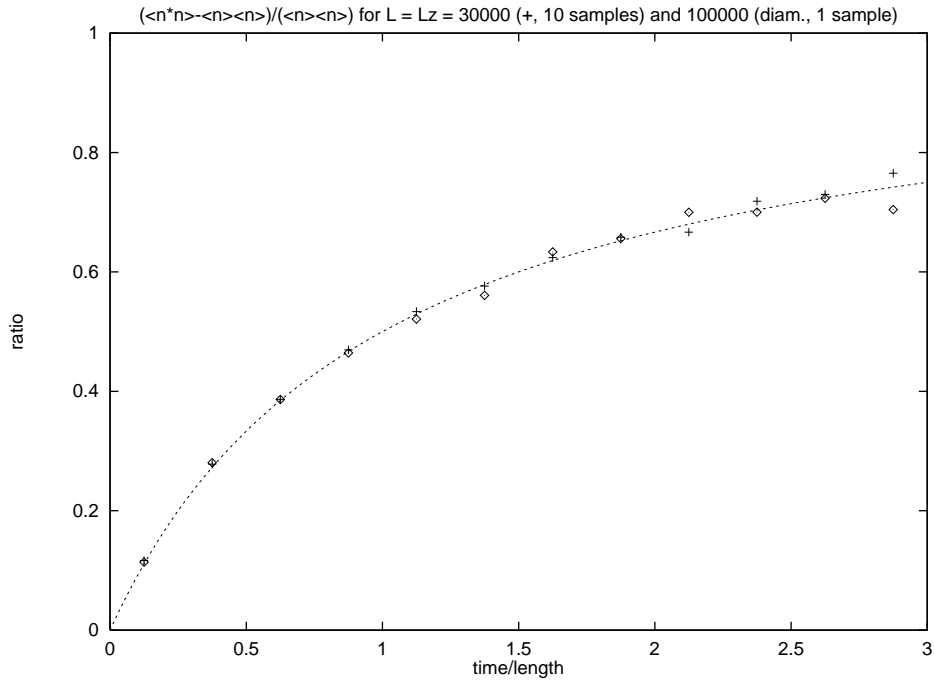


Figure 7: Comparison of $\langle (n - \langle n \rangle)^2 \rangle / \langle n \rangle^2$ with theoretical time dependence $t/(t + L)$, where n is the length of a stretch (“run” [9]) of uninterrupted updatings of the same site.